1. [CLO 2] Answer the following questions:

(a) (5 points) Find the power set of {∅, {∅}}

Solution: P (A)={∅, {∅}, {{∅}}, {∅, {∅}} } .

(b) (5 points) Find the truth set for P (x): $x^3 \geq 1$ where the domain is the set of integers.

Solution: Note that if $x \leq 0$ then $x^3 \leq 0$. Therefore, the truth set cannot contain the negative numbers nor 0. Thus truth set for P (x) is {1, 2, 3, ...}.

(c) (5 points) Prove or disprove that for $\left\lceil \frac{x}{2} \right\rceil = \left\lfloor \frac{x+1}{2} \right\rfloor$ all real numbers *x*.

Solution: False: $\frac{1}{2}$ is a counterexample.

(d) (10 points) Using set identity laws show that $(A \cap B) \cup (A \cap \bar{B}) = A$.

Solution: $(A \cap B) \cup (A \cap \bar{B}) = A \cap (B \cup \bar{B})$ by distribution law

$= A \cap (U)$ by complement law

$= A$ by Identity law

2. [CLO 2] Answer the following questions:

(a) (5 points). Show that $\sum_{j=1}^{n} a_j - a_{j-1} = a_n - a_0$ where $a_0, a_1, a_2, \cdots, a_n$ is a sequence of real numbers.

Solution: By expanding the summation, we get:

$$\sum_{j=1}^{n} a_j - a_{j-1} = (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1})$$

$$= (\cancel{a_1} - a_0) + (\cancel{a_2} \cancel{- a_1}) + (\cancel{a_3} \cancel{- a_2}) + \cdots + (\cancel{a_{n-1}} \cancel{-} \cancel{a_{n-2}}) + (a_n \cancel{- a_{n-1}})$$

$$(-a_0) + (a_n) = a_n - a_0$$

(b) (10 points) [CLO 2] Prove that if A and B are disjoint, countable, infinite sets, then A ∪ B is also countable.

Solution:

Because both *A* and *B* are countably infinite, we can list their elements as $a_1, a_2, \cdots, a_n, \cdots$ and $b_1, b_2, \cdots, b_n, \cdots$ respectively. By alternating terms of these two sequences, we can list the elements of *A* ∪ *B* in the infinite sequence *a*1, $a_1, b_1, a_2, b_2, \cdots, a_n, b_n, \cdots$. This means *A* ∪ *B* must be countably infinite.

(c) (10 points) [CLO 2] Suppose that g is a function from A to B and f is a function from B to C, then show that if both f and g are one-to-one functions then f ∘ g is also one-to-one.

Solution: Let x and y be distinct elements of A. Because g is one-to-one, g(x) and g(y) are distinct elements of B. Because f is one-to-one, f(g(x)) = (f ∘ g)(x) and f(g(y)) = (f ∘ g)(y) are distinct elements of C. Hence, f ∘ g is one-to-one.

3. (15 points) [CLO 2] Using mathematical induction prove that $n^2 - 7n + 12$ is nonnegative whenever n is an integer with $n \geq 3$.

Solution: **Basis step**: Let n = 3. Then

$$n^2 - 7n + 12 = 3^2 - 7 \cdot 3 + 12 = 9 - 21 + 12 = 0.$$

Inductive hypothesis: Assume for some integer $k \geq 3$ that $k^2 - 7k + 12$ is nonnegative.

Inductive step:

$$\begin{aligned}(k+1)^2 - 7(k+1) + 12 &= k^2 + 2k + 1 - 7k - 7 + 12 \\ &= (k^2 - 7k + 12) + (2k + 1 - 7) \\ &\geq \quad 0 \qquad\qquad + 2k + 1 - 7 \\ &= 2k - 6 \\ &\geq 2 \cdot 3 - 6 = 0\end{aligned}$$

4. [CLO 1] Give a recursive definition of:

(a) (5 points) The set of even integers

Solution: Basis Step: $0 \in S$, Recursive Step: If $x \in S$, then $x + 2 \in S$ and $x - 2 \in S$.

(b) (5 points) The set of positive integers not divisible by 5

Solution: Basis Step: $1 \in S$, $2 \in S$, $3 \in S$, $4 \in S$ Recursive Step: If $x \in S$, then $x + 5 \in S$.

5. Answer all of the following questions:

(a) (15 points) [CLO 3] How many bit string of length eight either start with a 1 bit or end with a two bits 00?

Solution: The number of bit strings either start with a 1 bit or end with a two bits 00 is equal to All the bit strings that start with 1 + all the bit strings that ends with 00 – all the bit strings that start with 1 and ends with 00.

# of 1xxxxxxx + # of xxxxxx00 – # of 1xxxxx00

$\mathbf{2^7 + 2^6 - 2^5}$

(b) (10 points) [CLO 3] Show that if five integers are selected from the first eight positive integers {1, 2, 3, 4, 5, 6, 7, 8}, there must be a pair of these integers with a sum equal to 9.

Solution: Group the first eight positive integers {1, 2, 3, 4, 5, 6, 7, 8} into four subsets of two integers each so that the integers of each subset add up to 9: {1, 8}, {2, 7}, {3, 6}, and {4, 5}. If five integers are selected from the first eight positive integers, by the pigeonhole principle at least two of them come from the same subset. Two such integers have a sum of 9, as desired.